Homework 6 - Solutions

(1) $A = \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}$ Row Reduce!

$$\xrightarrow{-\frac{1}{4}R_1} \begin{bmatrix} 1 & 3/4 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix} \xrightarrow[-5R_1+R_4]{-R_1+R_3} \begin{bmatrix} 1 & 3/4 & 0 \\ 0 & -1 & 4 \\ 0 & -3/4 & 3 \\ 0 & 1/4 & 6 \end{bmatrix}$$

$$\xrightarrow{-1R_2} \begin{bmatrix} 1 & 3/4 & 0 \\ 0 & 1 & -4 \\ 0 & -3/4 & 3 \\ 0 & 1/4 & 6 \end{bmatrix} \xrightarrow[\frac{3}{4}R_2+R_3,\ -\frac{1}{4}R_2+R_4]{-\frac{3}{4}R_2+R_1} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{7}R_4} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[4R_4+R_2]{-3R_4+R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\text{exchange } R_3 \,\&\, R_4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

every column has a pivot (the leading 1)

thus the columns of A are linearly independent.

(2)

a) $\left\{ \begin{bmatrix} a \\ a \\ a \\ a \end{bmatrix} : a \in \mathbb{R} \right\} = \left\{ a \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

the set $\boxed{\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}}$ is linearly independent and thus a basis

b) $\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a+b+c+d=0 \right\}$

let $a = -b-c-d$ then $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -b-c-d \\ b \\ c \\ d \end{bmatrix} = b \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

the set $\boxed{\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$ is linearly independent and a basis for the space

(3) Row Reduce A!

$$\begin{bmatrix} 2 & 5 & -8 & 7 \\ -1 & 5 & 4 & 7 \\ 0 & 5 & 0 & 7 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 5/2 & -4 & 7/2 \\ -1 & 5 & 4 & 7 \\ 0 & 5 & 0 & 7 \end{bmatrix} \xrightarrow{R_1+R_2} \begin{bmatrix} 1 & 5/2 & -4 & 7/2 \\ 0 & 15/2 & 0 & 21/2 \\ 0 & 5 & 0 & 7 \end{bmatrix}$$

$$\xrightarrow{2/15 R_2} \begin{bmatrix} 1 & 5/2 & -4 & 7/2 \\ 0 & 1 & 0 & 7/5 \\ 0 & 5 & 0 & 7 \end{bmatrix} \xrightarrow[-5R_2+R_3]{-5/2R_2+R_1} \begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 0 & 7/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\quad a_1 \quad a_2 \quad a_3 \quad a_4$

$a_3 = -4a_1$
$a_4 = 7/5\, a_2$ thus we can discard a_3, a_4

basis $= \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} \right\}$

④ Claim: A is an $n \times n$ matrix and λ is an eigenvalue value of A. Then 2λ is an eigenvalue of $2A$.

Pf: since λ is an eigenvalue of A, then there exists a non-zero vector x in $\mathbb{R}^n$ such that $Ax = \lambda x$

Multiplying both sides of the equation by 2 results in

$$2(Ax) = 2(\lambda x)$$
$$\Rightarrow (2A)x = (2\lambda)x$$
$\Rightarrow$ 2λ is an eigenvalue of $2A$. ∎